ON THE TOPOLOGICAL FULL GROUP OF A MINIMAL CANTOR $\mathbf{Z}^2$-SYSTEM

GÁBOR ELEK AND NICOLAS MONOD

Abstract. Grigorchuk and Medynets recently announced that the topological full group of a minimal Cantor $\mathbf{Z}$-action is amenable. They asked whether the statement holds for all minimal Cantor actions of general amenable groups as well. We answer in the negative by producing a minimal Cantor $\mathbf{Z}^2$-action for which the topological full group contains a non-abelian free group.

1. Introduction

Let G be a group acting on a compact space Σ by homeomorphisms. The *topological full group* associated to this action is the group of all homeomorphisms of Σ that are piecewise given by elements of G, each piece being open. Thus there are finitely many pieces at a time, all are clopen, and this construction is most interesting when Σ is a Cantor space. The importance of the topological full group has come to the fore in the classification results of Giordano–Putnam–Skau [2, 3].

Grigorchuk and Medynets announced that the topological full group of a minimal Cantor $\mathbf{Z}$-action is amenable [6]. This is particularly interesting in combination with the work of Matui [8], who showed that the derived subgroup is often a finitely generated simple group. Grigorchuk–Medynets further asked in [6] whether their result holds for actions of general amenable groups as well. We shall prove that it fails already for the group $\mathbf{Z}^2$:

Theorem 1. *There exists a free minimal Cantor $\mathbf{Z}^2$-action whose topological full group contains a non-abelian free group.*

Three comments are in order, see the end of this note:

1. There also exist free minimal Cantor $\mathbf{Z}^2$-actions whose topological full group is amenable, indeed locally virtually abelian.

2. Minimality is fundamental for the study of topological full groups. Even for $\mathbf{Z}$, it is easy to construct Cantor systems whose topological full group contains a non-abelian free group (using e.g. ideas from [9] or [4]).

3. Our example will be a minimal subshift and in this situation the topological full group is sofic by a result of [1].

2. Proof of the Theorem

We realize the Cantor space as the space Σ of all proper edge-colourings of the "quadrille paper" two-dimensional Euclidean lattice by the letters A, B, C, D, E, F (with the topology of pointwise convergence relative to the discrete topology on the finite set of letters). Recall here

Work supported in part by a Marie Curie grant, the European Research Council and the Swiss National Science Foundation.

that an edge-colouring is called *proper* if the edges adjacent to a given vertex are coloured differently. There is a natural $\mathbf{Z}^2$-action on Σ by homeomorphisms defined by translations.

To each letter $x \in \{A, \ldots, F\}$ corresponds a continuous involution of Σ, which we still denote by the same letter. It is defined as follows on $\sigma \in \Sigma$: if the vertex zero is connected to one of its four neighbours v by an edge labelled by x, then v is uniquely determined and $x\sigma$ will be the colouring σ translated towards v (i.e. the origin is now where v was). Otherwise, $x\sigma = \sigma$. This involution is contained in the topological full group of the $\mathbf{Z}^2$-action.

We have thus a homomorphism from the free product $\langle A\rangle * \cdots * \langle F\rangle$ to the topological full group. Notice that this free product preserves any $\mathbf{Z}^2$-invariant subset of Σ. We shall establish Theorem 1 by proving that Σ contains a minimal non-empty closed $\mathbf{Z}^2$-invariant subset M on which the $\mathbf{Z}^2$-action is free and on which the action of $\Delta := \langle A\rangle * \langle B\rangle * \langle C\rangle$ is faithful. This implies the theorem indeed, for Δ has a (finite index) non-abelian free subgroup.

A *pattern* of a colouring $\sigma \in \Sigma$ is the isomorphism class of a finite labelled subgraph of σ. We call σ *homogeneous* if for any pattern P of σ there is a number $f(P)$ such that the $f(P)$-neighbourhood of any vertex in the lattice contains the pattern P. The following facts are well-known and elementary (see e.g. [5]).

Lemma 2. *The orbit closure of $\sigma \in \Sigma$ is minimal if and only if σ is homogeneous. In that case, any τ in the orbit closure has the same patterns as σ and is homogenous with the same function f.* □

Now, we first enumerate the non-trivial elements of the free product Δ. Then, we label the integers with the natural numbers in such a way that the following property holds: for each $i \in \mathbf{N}$ there is $g(i) \geq 1$ such that any subinterval of length $g(i)$ in $\mathbf{Z}$ contains at least one element labelled by i. Such a labelling exists: for instance, label an integer by the exponent of 2 in its prime factorization (with an arbitrary adjustment for 0).

We use the labelling above to construct a specific proper edge-colouring $\lambda \in \Sigma$. Let w be a word in Δ that is the i-th in the enumeration. Consider the vertical vertex-lines $(v, \cdot)$ in the lattice such that v is labelled by i. Colour those vertical lines the following way. Starting at the point $(v, 0)$, copy the string w onto the half-line above, beginning from the right end of w (i.e. write w^{-1} upwards). Then colour the following edge by D, then copy the string w again and repeat the process ad infinitum. Also, continue the process below $(v, 0)$ so as to obtain a periodic colouring of the whole vertical line. Repeating the process for all non-trivial words w, we have coloured all vertical lines. Finally, colour all horizontal lines periodically with E and F.

The resulting colouring λ has the following property. For any non-trivial $w \in \Delta$ there is a number $h(w)$ such that the $h(w)$-neighbourhood of any vertex of the lattice contains a vertical string of the form $w^{-1}D$. Let $\Omega(\lambda) \subseteq \Sigma$ be the $\mathbf{Z}^2$-orbit closure of λ. Then all the elements of $\Omega(\lambda)$ have the same property. Now, let M be an arbitrary minimal subsystem of $\Omega(\lambda)$ (in fact it is easy to see that λ is homogeneous and hence $\Omega(\lambda)$ is already minimal). Notice that the $\mathbf{Z}^2$-action on M is free because λ has no period. In order to prove the theorem, it is enough to show that for any σ in M and any non-trivial $w \in \Delta$ there exists a $\mathbf{Z}^2$-translate of σ which is not fixed by w.

Pick thus any $\sigma \in M$. Then, by the above property of the orbit closure, there exists a translate τ of σ such that the vertical half-line pointing upwards from the origin starts with

the string $w^{-1}D$. Hence if we apply w to the translate we reach a point τ such that the colour of the edge pointing upwards from the the origin is coloured by D. Thus τ is not fixed by w, finishing the proof. $\square$

3. Comments

Some $\mathbf{Z}^2$-systems have a completely opposite behaviour to the ones constructed for Theorem 1. We shall see this by extending the method of Proposition 2.1 in [7].

Recall that the *p-adic odometer* is the minimal Cantor system given by adding 1 in the ring $\mathbf{Z}_p$ of p-adic integers. Taking the direct product, we obtain a minimal Cantor $\mathbf{Z}^2$-action on $\Sigma := \mathbf{Z}_p \times \mathbf{Z}_p$. The proposition below and its proof can be immediately extended to products of more general odometers.

Proposition 3. *The full group of this minimal Cantor $\mathbf{Z}^2$-system is an increasing union of virtually abelian groups.*

Proof (compare [7]*).* Consider $\mathbf{Z}_p$ as the space of $\mathbf{Z}/p\mathbf{Z}$-valued (infinite) sequences. Given a pair of finite sequences of length n, we obtain an n-cylinder set in Σ as the space of pairs of sequences starting with the given prefixes. Thus, n-cylinders determine a partition $\mathscr{P}_n$ of Σ into p^{2n} clopen subsets. Moreover, the clopen partition associated to any given element g of the topological full group can be refined to $\mathscr{P}_n$ when n is large enough. It remains only to observe that the collection of all such g, when n is fixed, is a subgroup of the semi-direct product $(\mathbf{Z}^2)^{\mathscr{P}_n} \rtimes \mathrm{Sym}(\mathscr{P}_n)$, where $\mathrm{Sym}(\mathscr{P}_n)$ is the permutation group of the coördinates indexed by $\mathscr{P}_n$. $\square$

Regarding the second comment of the introduction, suffice it to say that a *generic* proper colouring of the linear graph by three letters A, B, C gives a faithful non-minimal representation of the free product $\langle A \rangle * \langle B \rangle * \langle C \rangle$ into the topological full group of the associated $\mathbf{Z}$-subshift (compare [9] or [4] for generic constructions).

As for the last comment, Proposition 5.1(1) in [1] implies that the topological full group of any minimal subshift of any amenable group is a sofic group (in the notations of [1], the kernel N_Γ is trivial by an application of Lemma 2). In combination with Matui's results [8], this already shows the existence of a sofic finitely generated infinite simple group without appealing to [6].

References

[1] G. Elek and E. Szabó. Hyperlinearity, essentially free actions and L^2-invariants. The sofic property. *Math. Ann.*, 332(2):421–441, 2005.

[2] T. Giordano, I. F. Putnam, and C. F. Skau. Topological orbit equivalence and C^*-crossed products. *J. Reine Angew. Math.*, 469:51–111, 1995.

[3] T. Giordano, I. F. Putnam, and C. F. Skau. Full groups of Cantor minimal systems. *Israel J. Math.*, 111:285–320, 1999.

[4] Y. Glasner and N. Monod. Amenable actions, free products and a fixed point property. *Bull. Lond. Math. Soc.*, 39(1):138–150, 2007.

[5] W. H. Gottschalk. Almost period points with respect to transformation semi-groups. *Ann. of Math. (2)*, 47:762–766, 1946.

[6] R. I. Grigorchuk and K. Medynets. On simple finitely generated amenable groups. Preprint, `http://arxiv.org/abs/math/1105.0719v2`.

[7] H. Matui. Some remarks on topological full groups of Cantor minimal systems II. Preprint, `http://arxiv.org/abs/math/1111.3134v1`.

[8] H. Matui. Some remarks on topological full groups of Cantor minimal systems. *Internat. J. Math.*, 17(2):231–251, 2006.

[9] E. K. van Douwen. Measures invariant under actions of F_2. *Topology Appl.*, 34(1):53–68, 1990.

EPFL, 1015 Lausanne, Switzerland